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Markov Inequalities for Weight Functions of Chebyshev Type*

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Denote by $\eta_i = \cos(i\pi/n)$, $i = 0, ..., n$ the extreme points of the Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$. Let π_n be the set of real algebraic polynomials of degree not exceeding n, and let B_n be the unit ball in the space π_n equipped with the discrete norm $|p|_{n,\infty} := \max_{0 \leqslant i \leqslant n} |p(\eta_i)|$. We prove that the unique solutions of the extremal problems $\max_{p \in B_n} \int_{-1}^{1} [p^{(k+1)}(x)]^2 (1-x^2)^{k-1/2}\, dx$, $k = 0, ..., n-1$, and $\max_{p \in B_n} \int_{-1}^{1} [p^{(k+2)}(x)]^2 (1-x^2)^{k-1/2}\, dx$, $k = 0, ..., n-2$, are $p(x) = \pm T_n(x)$, and we obtain the extremal values in an explicit form. © 1995 Academic Press, Inc.

1. INTRODUCTION

Let $\|\cdot\|_\infty$ be the uniform norm on $[-1, 1]$. In 1941 Duffin and Schaeffer [2] proved that if $p \in B_n$ then

$$\|p^{(k)}\|_\infty \leqslant \frac{n^2(n^2-1^2)\cdots(n^2-(k-1)^2)}{1.3\cdots(2k-1)}, \qquad k = 1, ..., n, \tag{1}$$

and the bounds are attained only for $p = \pm T_n$. This result is a refinement of a theorem of Markov [3] who proved (1) under the stronger requirement $\|p\|_\infty \leqslant 1$.

Let $\|f\|_q := [\int_{-1}^{1} |f(x)|^q\, dx]^{1/q}$, $1 \leqslant q < \infty$. It was proved in [1] that for any q, $1 \leqslant q < \infty$, and every $p \in \pi_n$:

$$\|p'\|_q \leqslant \|T_n'\|_q\, \|p\|_\infty. \tag{2}$$

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In this paper we establish a weighted analogue of (1) and (2) for $q=2$ in the sense that the L_2 norms of $p^{(k)}$, $k=1, ..., n$, for some specific weights are compared with the $|\cdot|_{n,\infty}$ norm of p. The result reads as follows.

THEOREM 1. *For every $p \in \pi_n$ the inequalities*

$$\int_{-1}^{1} [p^{(k+1)}(x)]^2 (1-x^2)^{k-1/2}\, dx \leqslant \frac{\pi n^2}{2k+1} \frac{(n+k)!}{(n-k-1)!} |p|^2_{n,\infty}, \qquad k=0, ..., n-1, \tag{3}$$

and

$$\int_{-1}^{1} [p^{(k+2)}(x)]^2 (1-x^2)^{k-1/2}\, dx \leqslant \frac{2\pi n^2}{(2k+1)(2k+3)} \frac{(n+k+1)!}{(n-k-2)!} \left(\frac{n^2-(k+2)^2}{2k+5} + k + 1\right) |p|^2_{n,\infty}, \qquad k=0, ..., n-2, \tag{4}$$

hold. Equalities are attained if and only if $p(x) = cT_n(x)$, where c is an arbitrary real constant.

Note that in the particular case $k=0$ inequality (4) is due to Varma [6]. Actually, it was proved in [6, Theorem 2] that if $p \in \pi_n$ and $\|p\|_\infty \leqslant 1$ then

$$\int_{-1}^{1} [p''(x)]^2 (1-x^2)^{-1/2}\, dx \leqslant \int_{-1}^{1} [T''_n(x)]^2 (1-x^2)^{-1/2}\, dx.$$

2. PROOF

The proof of the theorem is preceded by three lemmas. The first one summarizes Lemmas 1 and 4 and Theorem 3 in [2].

LEMMA 1. *Let $p \in \pi_n$ and $|p|_{n,\infty} \leqslant 1$. Then the inequalities*

$$|p^{(k)}(\pm 1)| \leqslant |T_n^{(k)}(\pm 1)| = \frac{n^2(n^2-1^2)\cdots(n^2-(k-1)^2)}{1.3\cdots(2k-1)}, \qquad k=0, ..., n, \tag{5}$$

and

$$|p^{(k+1)}(x_i)| \leqslant |T_n^{(k+1)}(x_i)|, \qquad \text{whenever} \quad T_n^{(k)}(x_i) = 0, \qquad k = 0, ..., n-1, \tag{6}$$

hold with equalities only for $p = \pm T_n$.

Lemma 2 concerns Gaussian and generalized Lobatto quadrature formulae associated with the Gegenbauer weight functions $\omega_\lambda(x) := (1-x^2)^{\lambda - 1/2}$ for nonnegative integer values of λ. Let C_n^λ, $n = 0, 1, ...;$ $\lambda > -1/2$, be the Gegenbauer polynomials, orthogonal on $[-1, 1]$ with respect to ω_λ, and let $x_{i,n}^{(\lambda)}$, $i = 1, ..., n$, be the zeros of C_n^λ. By $\mu_{i,n}^{(\lambda)}$ we mean the Cotes numbers of the Gaussian quadrature formula

$$\int_{-1}^{1} f(x)\, \omega_\lambda(x)\, dx \approx \sum_{i=1}^{n} \mu_{i,n}^{(\lambda)} f(x_{i,n}^{(\lambda)}),$$

associated with ω_λ, which has the algebraic degree of precision $2n-1$.

For every pair of natural numbers l and n there exists a unique quadrature rule of the form

$$\begin{aligned}\int_{-1}^{1} f(x)\, \omega_\lambda(x)\, dx &\approx \sum_{j=0}^{l-1} a_j(\lambda, l, n)(f^{(j)}(-1) + (-1)^j f^{(j)}(1)) \\ &\quad + \sum_{i=1}^{n} \mu_i(\lambda, l, n) f(x_i) \\ &=: Q(f; \lambda, l, n),\end{aligned}$$

which is precise for every polynomial of degree $2n + 2l - 1$. It is called the generalized Lobatto quadrature formula. It is easily seen that $a_j(\lambda, l, n) > 0$ and

$$\mu_i(\lambda, l, n) = (1 - x_i^2)^{-l} \mu_{i,n}^{(\lambda + l)} > 0. \tag{7}$$

The nodes of $Q(f; \lambda, l, n)$ are located at the zeros of $C_n^{\lambda + l}$; i.e.,

$$x_i = x_{i,n}^{(\lambda + l)}, \qquad i = 1, ..., n. \tag{8}$$

Lemma 2. *For any given* n *and* k, $0 \leqslant k \leqslant n$, *let* $\xi_{i,n}^{(k)}$, $i = 1, ..., n-k$, *be the zeros of* $T_n^{(k)}$. *Then the quadrature formulae*

$$\int_{-1}^{1} f(x)\,\omega_k(x)\,dx \approx \sum_{i=1}^{n-k} \mu_{i,n-k}^{(k)} f(\xi_{i,n}^{(k)}), \qquad 0 \leqslant k \leqslant n-1, \tag{9}$$

$$\int_{-1}^{1} f(x)\,\omega_k(x)\,dx \approx a_0(k, 1, n-k-1)(f(-1)+f(1)) + \sum_{i=1}^{n-k-1} \mu_i(k, 1, n-k-1) f(\xi_{i,n}^{(k+1)}), \qquad 0 \leqslant k \leqslant n-2, \tag{10}$$

and

$$\int_{-1}^{1} f(x)\,\omega_k(x)\,dx \approx a_0(k, 2, n-k-2)(f(-1)+f(1)) + a_1(k, 2, n-k-2)(f'(-1)-f'(1)) + \sum_{i=1}^{n-k-2} \mu_i(k, 2, n-k-2) f(\xi_{i,n}^{(k+2)}), \qquad 0 \leqslant k \leqslant n-3, \tag{11}$$

have algebraic degree of precision $2n-2k-1$. *Moreover,*

$$a_0(k, 1, n-k-1) = 2^{2k-1}(2k+1)\,\Gamma^2(k+1/2)\,\frac{(n-k-1)!}{(n+k)!} \tag{12}$$

and

$$a_1(k, 2, n-k-2) = 2^{2k}(2k+3)\,\Gamma^2\left(k+\frac{3}{2}\right)\frac{(n-k-2)!}{(n+k+1)!}, \tag{13}$$

$$a_0(k, 2, n-k-2) = \frac{a_1(k, 2, n-k-2)}{2k+1} \times \left(\frac{2(n^2-(k+2)^2)(2k+3)}{2k+5} + 4(k+1)\right). \tag{14}$$

Proof. It is well known that T_n, $n=0, 1, \ldots$, are orthogonal on $[-1, 1]$ with respect to $\omega_0(x) = (1-x^2)^{-1/2}$. Hence $T_n(x) = c_1 C_n^0(x)$ (here and in what follows by c_i we mean nonzero constants). On the other hand [5, Chap. 4.7],

$$\frac{d}{dx} C_n^{\lambda}(x) = c_2 C_{n-1}^{\lambda+1}(x)$$

and then

$$\frac{d^k}{dx^k} C_n^{\lambda}(x) = c_3 C_{n-k}^{\lambda+k}(x).$$

Applying the latter for $\lambda = 0$ we get $T_n^{(k)}(x) = c_4 C_{n-k}^{k}(x)$, which yields

$$\xi_{i,n}^{(k)} = x_{i,n-k}^{(k)}, \qquad i = 1, ..., n-k. \tag{15}$$

Therefore, $\xi_{i,n}^{(k)}$, $i = 1, ..., n-k$, are the nodes of the Gaussian quadrature with $n-k$ nodes associated with ω_k.

Taking into account the relation (8) between the nodes of the Gaussian and Lobatto's rules and applying (15) for $k+1$ we conclude that Lobatto's formula associated with ω_k has $\xi_{i,n}^{(k+1)}$ for its inside nodes. Thus (10) coincides with $Q(f; k, 1, n-k-1)$. Similarly, the inside nodes of $Q(f; k, 2, n-k-2)$ are the zeros $\xi_{i,n}^{(k+2)}$, $i = 1, ..., n-k-2$, of $T_n^{(k+2)}$.

Explicit expressions for the coefficients a_{l-1} and a_{l-2} are given by Maskell and Sack [4, (3.9), (3.10)] even for generalized Lobato quadrature formulae associated with Jacobi weight functions.

LEMMA 3. *For any positive integer n we have*

$$\int_{-1}^{1} [T_n^{(k+1)}(x)]^2 (1-x^2)^{k-1/2}\, dx = \frac{\pi n^2}{2k+1} \frac{(n+k)!}{(n-k-1)!}, \qquad k = 0, ..., n-1, \tag{16}$$

and

$$\int_{-1}^{1} [T_n^{(k+2)}(x)]^2 (1-x^2)^{k-1/2}\, dx = \frac{2\pi n^2}{(2k+1)(2k+3)} \frac{(n+k+1)!}{(n-k-2)!} \left(\frac{n^2-(k+2)^2}{2k+5} + k + 1 \right), \qquad k = 0, ..., n-2. \tag{17}$$

Proof. Lobatto's formula (10) is precise for $f = [T_n^{(k+1)}]^2$. Hence by means of (5) and (12) we get

$$\begin{aligned}
\int_{-1}^{1} & [T_n^{(k+1)}(x)]^2 (1-x^2)^{k-1/2}\, dx \\
&= 2a_0(k, 1, n-k-1)[T_n^{(k+1)}(1)]^2 \\
&= 2^{2k}(2k+1)\, \Gamma^2\left(k+\frac{1}{2}\right) \frac{(n-k-1)!}{(n+k)!} \left[\frac{n^2(n^2-1^2)\cdots(n^2-k^2)}{1.3\cdots(2k+1)} \right]^2.
\end{aligned}$$

Using the recurrence relation for the gamma function and $\Gamma(\frac{1}{2})=\sqrt{\pi}$ we obtain (16).

In order to establish (17) we apply (11) to $f=[T_n^{(k+2)}]^2$:

$$\begin{aligned}\int_{-1}^{1} & [T_n^{(k+2)}(x)]^2\,(1-x^2)^{k-1/2}\,dx\\ &=2a_0(k,2,n-k-2)[T_n^{(k+2)}(1)]^2\\ &\quad+2a_1(k,2,n-k-2)[T_n^{(k+2)}(-1)\,T_n^{(k+3)}(-1)\\ &\quad-T_n^{(k+2)}(1)\,T_n^{(k+3)}(1)]\\ &=2a_0(k,2,n-k-2)[T_n^{(k+2)}(1)]^2\\ &\quad-4a_1(k,2,n-k-2)\,T_n^{(k+2)}(1)\,T_n^{(k+3)}(1).\end{aligned}$$

From $T_n^{(k+3)}(1)=((n^2-(k+2)^2)/(2k+5))\,T_n^{(k+2)}(1)$ and (14) we get

$$\begin{aligned}\int_{-1}^{1} & [T_n^{(k+2)}(x)]^2\,(1-x^2)^{k-1/2}\,dx\\ &=2a_1(k,2,n-k-2)[T_n^{(k+2)}(1)]^2\\ &\quad\times\left[\frac{1}{2k+1}\left(\frac{2(n^2-(k+2)^2)(2k+3)}{2k+5}+4(k+1)\right)-2\,\frac{n^2-(k+2)^2}{2k+5}\right]\\ &=8a_1(k,2,n-k-2)[T_n^{(k+2)}(1)]^2\left[\frac{n^2-(k+2)^2}{(2k+1)(2k+5)}+\frac{k+1}{2k+1}\right].\end{aligned}$$

The formulae (13) and (5) yield (17).

Proof of the theorem. Let $p\in B_n$. Then the inequalities (6) are equivalent to

$$|p^{(k+1)}(\xi_{i,n}^{(k)})|\leqslant|T_n^{(k+1)}(\xi_{i,n}^{(k)})|,\qquad i=1,\ldots,n-k. \tag{18}$$

Since $[p^{(k+1)}]^2\in\pi_{2n-2k-2}$, (9) has algebraic degree of precision $2n-2k-1$ and the Cotes numbers are positive, then

$$\begin{aligned}\int_{-1}^{1}[p^{(k+1)}(x)]^2\,\omega_k(x)\,dx &=\sum_{i=1}^{n-k}\mu_{i,n-k}^{(k)}\,[p^{(k+1)}(\xi_{i,n}^{(k)})]^2\\ &\leqslant\sum_{i=1}^{n-k}\mu_{i,n-k}^{(k)}[T_n^{(k+1)}(\xi_{i,n}^{(k)})]^2\\ &=\int_{-1}^{1}[T_n^{(k+1)}(x)]^2\,\omega_k(x)\,dx.\end{aligned}$$

Now inequalities (3) follow from (16).

The inequality

$$\int_{-1}^{1} [p^{(k+2)}(x)]^2 \omega_k(x)\, dx \leqslant \int_{-1}^{1} [T_n^{(k+2)}(x)]^2 \omega_k(x)\, dx, \qquad p \in B_n,$$

can be established in a similar way. One applies (10) to $[p^{(k+2)}]^2$ and then, having in mind (7) and (12), use (18) for $k+1$ and (5) for $k+2$, respectively.

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